Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 12, 2002*
Accepted: *August 30, 2002*

Two-loop self-dual Euler-Heisenberg lagrangians 1. Real part and helicity amplitudes

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Abstract: We show that, for both scalar and spinor QED, the two-loop Euler-Heisenberg effective lagrangian for a constant euclidean self-dual background has an extremely simple closed-form expression in terms of the digamma function. Moreover, the scalar and spinor QED effective lagrangians are very similar to one another. These results are dramatic simplifications compared to the results for other backgrounds. We apply them to a calculation of the low energy limits of the two-loop massive N-photon 'all +' helicity amplitudes. The simplicity of our results can be related to the connection between self-duality, helicity and supersymmetry.

Keywords: Renormalization Regularization and Renormalons, Nonperturbative Effects, Electromagnetic Processes and Properties.

Contents

1. Introduction: QED and QCD in constant fields

The Euler-Heisenberg lagrangian, one of the earliest results in quantum electrodynamics [1, 2], describes the complete one-loop amplitude involving a spinor loop interacting non-perturbatively with a constant background electromagnetic field. Euler and Heisenberg found the following well-known integral representation for this effective lagrangian:

$$\mathcal{L}^{(1)}_{\rm spin} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\left[\frac{e^2ab}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2) - \frac{1}{T^2}\right]. \tag{1.1}$$

Here T denotes the (euclidean) propertime of the loop fermion, and a, b are related to the two invariants of the Maxwell field by $a^2 - b^2 = B^2 - E^2, ab = \mathbf{E}\cdot\mathbf{B}$. The subtractions of the terms of zeroeth and second order in a, b corresponds to on-shell renormalization.

The Euler-Heisenberg lagrangian contains the information on the low — enery limit of the one-loop N-photon amplitudes for any N. Moreover, it does so in a form which is extremely convenient for the study at low energies of nonlinear QED effects [3] such as photon-photon scattering [1], photon dispersion [4, 5], and photon splitting [6, 5, 7, 8]. For scalar QED, an analogous result was obtained by Schwinger [9]:

$$\mathcal{L}^{(1)}_{\rm scal} = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\left[\frac{e^2ab}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2) - \frac{1}{T^2}\right]. \tag{1.2}$$

The lagrangians (1.1) and (1.2) are real for a purely magnetic field, while in the presence of an electric field there is an absorptive part, indicating the process of electron-positron (resp. scalar-antiscalar) pair creation by the field [9].

The first radiative corrections to these lagrangians, describing the effect of an additional photon exchange in the loop, were obtained in the seventies by Ritus [10, 11, 12]. Using the exact propagators in a constant field found by Fock [13] and Schwinger [9], and a proper-time cutoff as the UV regulator, Ritus obtained the corresponding two-loop effective lagrangians $\mathcal{L}^{(2)}_{\rm scal,spin}$ in terms of certain two-parameter integrals. Similar two-parameter integral representations for $\mathcal{L}^{(2)}_{\rm scal,spin}$ where obtained later by other authors, using either proper-time [14, 15] or dimensional regularisation [16, 17]. Unfortunately, all of these double parameter integral representations are quite complicated, so that it is much more difficult to study the weak- and strong-field expansions at two-loops than at one-loop. Even more difficult becomes the analysis of the imaginary part of the effective action [18]. This is true even for the special cases where the background is purely magnetic or purely electric [11, 12].

However, the magnetic/electric cases are not the simplest ones which one can study in this context. As we will see in the following, the constant field background which leads to maximal simplification is the (euclidean) self-dual one, given by[1]

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\,. \tag{1.3}$$

This is because for such a field the square of the field strength tensor is proportional to the Lorentz identity (see eq. (2.2) below) so that the inevitable breaking of the Lorentz invariance by the background field is in some sense minimized. In Minkowski space the self-duality condition requires either E or B to be complex. However, this does not imply that such backgrounds are devoid of physical meaning. Rather, they should be interpreted in terms of helicity projections [20]. And indeed, it will be seen below that it is quite straightforward to extract, from the effective action for such a self-dual field, the low energy limit of the two-loop amplitude for the scattering of N photons with all equal helicities. In addition, we see at least four more good reasons for studying this particular self-dual case in detail.

First, a detailed comparison at the one-loop level shows that the self-dual case with real **B** and complex **E** (corresponding to a real self-dual field strength f, where $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and called the 'magnetic' case in the following) leads to an effective lagrangian whose properties are only marginally different from the ones of the magnetic Euler-Heisenberg lagrangian, while the case with a real electric and complex magnetic field (corresponding to f imaginary, and called the 'electric' case in the following) is a very good analogue of the electric case. This leads us to expect that the study of the self-dual case at higher loops may yield useful information on the generic properties of Euler-Heisenberg lagrangians.

[1]This euclidean self-duality condition should not be confused with the similar condition $a = \pm b$ which *can* be realized in Minkowski space and could also be called 'self-duality'. See [19] for a discussion of these various dualities.

Second, many quantities of physical interest in quantum electrodynamics are computable in euclidean space. This includes the renormalization constants, and in particular the QED β-function.

Third, the constant self-dual euclidean backgrounds that we consider here for QED generalize in a very simple way to the case of quasi-abelian self-dual constant backgrounds in QCD. Such backgrounds have been studied extensively in QCD as they have the special property [21] that among the covariantly constant gluon backgrounds, only the self-dual quasi-abelian background is stable (at one-loop) under fluctuations. This has led to extensive studies of quark and gluon loops in such a background [21, 22, 23, 24]. Also at one-loop, much is known about QCD in the presence of arbitrary covariantly constant background fields. One-loop results along the lines of the Euler-Heisenberg formulas (1.1) and (1.2) can be found in [25, 26]. The purely chromomagnetic covariantly constant background is unstable [27, 28, 21]. The renormalization of these one-loop results in pure QCD is complicated by infrared problems, problems which become even worse at the two-loop level [29].

A fourth motivation for studying constant quasi-abelian self-dual backgrounds is that they could provide useful information about effective lagrangians in other self-dual backgrounds, in particular instantons [30, 31, 32].

As our main result in this paper, we will show that in the self-dual case, at two-loop, *all* parameter integrals can be done in closed form, yielding the following simple expressions for the two-loop scalar and spinor QED effective lagrangians:

$$\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) = \alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\xi^2(\kappa) - \xi'(\kappa)\right] \tag{1.4}$$

$$\mathcal{L}_{\rm spin}^{(2)(SD)}(\kappa) = -2\alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[3\xi^2(\kappa) - \xi'(\kappa)\right]. \tag{1.5}$$

Here we have defined the convenient dimensionless parameter

$$\kappa \equiv \frac{m^2}{2e\sqrt{f^2}} \tag{1.6}$$

where $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, as well as the important function

$$\xi(x) \equiv -x\Big(\psi(x) - \ln(x) + \frac{1}{2x}\Big) \tag{1.7}$$

with ψ being the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. The scalar QED formula (1.4) was already presented in [33].

The extreme simplicity of these results, and the obvious similarity between the two-loop effective lagrangians in the scalar and spinor cases, provides a new motivation for studying self-dual backgrounds. Our results provide a new example of the well-known connections among self-duality, helicity and supersymmetry. Self-dual fields have definite helicity [20], and are closely related to supersymmetry [30, 34, 35]. One consequence of this connection is that there exist remarkably simple formulas for loop amplitudes when the external fields have all (or almost all) helicities being equal. Such simplifications have been known for a long time in massless QCD at the tree level [36]. More recently they have been studied extensively at the one- and two loop level, in abelian [37, 38, 39] as well as in nonabelian [40, 41, 42] gauge theory. The close interplay been self-duality, supersymmetry and integrability has also been explored in this context. The simplicity of the structure of the QED/QCD amplitudes with all (almost all) helicities alike is thought to be deeply related to the integrability properties of self-dual Yang-Mills fields [43, 44]. In the present paper the connection between self-duality and helicity will be used to obtain, from the above results for the self-dual two-loop effective lagrangians, the low energy limits of the QED N-photon amplitudes with all helicities alike.

The outline of the paper is as follows. In section 2 we write down the one-loop self-dual Euler-Heisenberg lagrangians for scalar and spinor QED, as well as for the 'quasi-abelian' special case in QCD. In section 3 we compute the two-loop corrections to these lagrangians for scalar and spinor QED, using the 'string-inspired' worldline formalism [45, 46] along the lines of [47, 48, 15, 16]. This formalism is based on the representation of effective actions in terms of first-quantized particle path integrals, and has turned out to be highly convenient for computations involving constant external fields [47, 49, 50, 51, 52, 15, 8, 16] (see [53] for a review). In section 4 we comment on the special properties of self-dual fields with respect to helicity and supersymmetry, and we use some of these properties to explain the similarity of our two-loop scalar and spinor QED results. We then proceed in section 5 to an explicit calculation of the low energy limits of the massive all '+' helicity N photon amplitudes, at one and two loops, in scalar and spinor QED. Section 6 contains a summary, and some possible future directions of work. In an accompanying paper [54] we study the weak- and strong-field expansions of these two-loop effective lagrangians, and use them to test the techniques of Borel summation, and the associated analytic continuation properties and structure of the imaginary part of the two-loop self-dual Euler-Heisenberg lagrangians for the 'electric' case.

2. One loop self-dual Euler-Heisenberg lagrangians

2.1 Scalar QED

For scalar QED, the (unrenormalized) one-loop effective lagrangian in a constant $F_{\mu\nu}$ background is [53]

$$\mathcal{L}^{(1)}_{\rm scal}(F) = (4\pi)^{-\frac{D}{2}} \int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \det{}^{-\frac{1}{2}} \left[\frac{\sin(eFT)}{eFT}\right] . \tag{2.1}$$

Here T is the proper-time variable for the loop scalar, and D is the spacetime dimension. We would like to evaluate this integral for the case of a self-dual field, $F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. The self-duality condition implies that

$$F^2 = -f^2 \mathbb{1} \tag{2.2}$$

where $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and $\mathbb{1}$ denotes the identity matrix in Lorentz space. Therefore

$$\det{}^{-\frac{1}{2}} \left[\frac{\sin(eFT)}{eFT}\right] = \Big(\frac{efT}{\sinh(efT)}\Big)^{\frac{D}{2}} . \tag{2.3}$$

Renormalization involves subtracting the free field contribution, and also a charge renormalization, corresponding to subtracting the logarithmically divergent $O(f^2)$ term. This leads to the following renormalized one-loop effective lagrangian:

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{4\kappa^2}\int_0^\infty \frac{dZ}{Z}\mathrm{e}^{-2\kappa Z}\left[\frac{1}{\sinh^2(Z)} - \frac{1}{Z^2} + \frac{1}{3}\right]. \tag{2.4}$$

We remark that this integral can be expressed in terms of the function $\ln\Gamma(x)$ and its integral function:

$$\mathcal{L}^{(1)(SD)}_{\rm scal}(\kappa) = \frac{m^4}{(4\pi)^2}\frac{1}{\kappa^2}\left[-\frac{1}{12}\ln(\kappa) + \zeta'(-1) + \Xi(\kappa)\right] \tag{2.5}$$

where the function $\Xi(x)$ is defined as follows:

$$\Xi(x) \equiv \int_0^x dy\, \ln\Gamma(y) - x\ln\Gamma(x) + \frac{x^2}{2}\ln(x) - \frac{x^2}{4} - \frac{x}{2}. \tag{2.6}$$

This one-loop result (2.5) may equivalently be expressed in terms of the Hurwitz zeta function (see, e.g., [14, 55]), but we prefer to use this representation here, since the two-loop result is a simple function of $\xi(\kappa) = \Xi'(\kappa)$.

2.2 Spinor QED

For spinor QED, the unrenormalized one-loop effective lagrangian in a constant $F_{\mu\nu}$ background is

$$\mathcal{L}^{(1)}_{\rm spin}(F) = -2(4\pi)^{-\frac{D}{2}}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}T^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\tan(eFT)}{eFT}\right]. \tag{2.7}$$

For a self-dual field

$$\det{}^{-\frac{1}{2}}\left[\frac{\tan(eFT)}{eFT}\right] = \left(\frac{efT}{\tanh(efT)}\right)^{\frac{D}{2}}. \tag{2.8}$$

After renormalization, this leads to the following effective lagrangian:

$$\mathcal{L}^{(1)(SD)}_{\rm spin}(\kappa) = -\frac{m^4}{(4\pi)^2}\frac{1}{2\kappa^2}\int_0^\infty \frac{dZ}{Z}\mathrm{e}^{-2\kappa Z}\left[\coth^2 Z - \frac{1}{Z^2} - \frac{2}{3}\right]. \tag{2.9}$$

2.3 QCD

In the nonabelian case, the natural notion of a constant field strength is covariant constancy, i.e. $\mathcal{D}^{ab}_\alpha F^b_{\mu\nu} = 0$. This does not imply that $F^b_{\mu\nu}$ is a constant matrix, so that further assumptions need to be made to arrive at Euler-Heisenberg type formulas. The simplest such case is the 'quasi-abelian' one, where $A_\mu = A^a_\mu T^a$, and therefore also $F_{\mu\nu}$, are assumed to point in a fixed direction in colour space:

$$A^a_\mu(x) = n^a A_\mu(x)\,, \qquad F^a_{\mu\nu}(x) = n^a F_{\mu\nu}(x)\,. \tag{2.10}$$

The computation of the one-loop effective action with such a field is identical to the abelian one for the scalar and spinor loop cases, since the colour degree of freedom manifests itself only in a global colour trace. Thus for the self-dual case one obtains (compare (2.1), (2.7))

$$\mathcal{L}^{(1)}_{\rm scal}(F) = {\rm tr}_c \int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Bigl(\frac{gfT\not{n}}{\sinh(gfT\not{n})}\Bigr)^{\frac{D}{2}} \tag{2.11}$$

$$\mathcal{L}^{(1)}_{\rm spin}(F) = -2\,{\rm tr}_c \int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Bigl(\frac{gfT\not{n}}{\tanh(gfT\not{n})}\Bigr)^{\frac{D}{2}}\,. \tag{2.12}$$

For the gluon loop one finds, in this quasi-abelian constant field case, the result [26, 55, 15]

$$\mathcal{L}^{(1)}_{\rm glu}(F) = \frac{1}{2}\,{\rm tr}_c \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \exp\Bigl[-\frac{1}{2}\,{\rm tr}_L \ln\Bigl(\frac{\sin(gFT)}{gFT}\Bigr)\Bigr]\,{\rm tr}_L \cos(2gFT)\,. \tag{2.13}$$

In these formulas, ${\rm tr}_L$ denotes the Lorentz trace, ${\rm tr}_c$ the colour trace, and we defined $\not{n} \equiv n^a T^a$. For the self-dual case, (2.13) can be written as

$$\mathcal{L}^{(1)}_{\rm glu}(F) = 2\,{\rm tr}_c \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\Bigl(\frac{gfT\not{n}}{\sinh(gfT\not{n})}\Bigr)^{\frac{D}{2}}\bigl(1+2\sinh^2(gfT\not{n})\bigr)\,. \tag{2.14}$$

Let us specialize further to the case where all particles are in the adjoint representation and massless. We can then combine the contribution of m (real) scalars, n (Weyl) spinors, the gauge boson and its ghost (which gives minus the contribution of a complex scalar) into (putting $D = 4$)

$$\mathcal{L}^{(1)}_{\rm total}(F) = \frac{1}{2}\,{\rm tr}_c \int_0^\infty \frac{dT}{T}(4\pi T)^{-2}\Bigl\{(8-2n)(gfT\not{n})^2 + (m-2n+2)\Bigl[\frac{gfT\not{n}}{\sinh(gfT\not{n})}\Bigr]^2\Bigr\}\,. \tag{2.15}$$

For $m = 6$ and $n = 4$ we recover the well-known fact that the one-loop self-dual effective action vanishes for $N = 4$ SYM theory [34, 20, 56].

3. Two loop self-dual Euler-Heisenberg lagrangians

3.1 Scalar QED

In [15, 16] the two-loop Euler-Heisenberg lagrangian in (euclidean) Scalar QED was obtained in terms of the following fourfold parameter integral,

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm scal}(F) = (4\pi)^{-D}\Bigl(-\frac{e^2}{2}\Bigr)\int_0^\infty \frac{dT}{T} e^{-m^2T} T^{-\frac{D}{2}} \int_0^\infty d\bar{T} \int_0^T d\tau_a \int_0^T d\tau_b \,\times \\ \times \det{}^{-\frac{1}{2}}\Bigl[\frac{\sin(eFT)}{eFT}\Bigr] \det{}^{-\frac{1}{2}}\Bigl[\bar{T} - \frac{1}{2}\mathcal{C}_{ab}\Bigr]\langle \dot{y}_a \cdot \dot{y}_b\rangle\,.\end{aligned} \tag{3.1}$$

Here T and $\bar{T}$ represent the scalar and photon proper-times, and $\tau_{a,b}$ the endpoints of the photon insertion moving around the scalar loop. $\mathcal{C}_{ab}$ and $\langle \dot{y}_a \cdot \dot{y}_b\rangle$ are given by

$$\begin{aligned}\mathcal{C}_{ab} &= \mathcal{G}_B(\tau_a,\tau_a) - \mathcal{G}_B(\tau_a,\tau_b) - \mathcal{G}_B(\tau_b,\tau_a) + \mathcal{G}_B(\tau_b,\tau_b) \\ \langle \dot{y}_a \cdot \dot{y}_b\rangle &= {\rm tr}\Bigl[\ddot{\mathcal{G}}_{Bab} + \frac{1}{2}\frac{(\dot{\mathcal{G}}_{Baa} - \dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab} - \dot{\mathcal{G}}_{Bbb})}{\bar{T} - \frac{1}{2}\mathcal{C}_{ab}}\Bigr]\,.\end{aligned} \tag{3.2}$$

They are expressed in terms of the worldline Green's function $\mathcal{G}_B$ and its first and second derivatives [15]:

$$
\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= \frac{1}{2(eF)^2}\left(\frac{eF}{\sin(eFT)}\mathrm{e}^{-ieFT\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{T}\right)\\
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \frac{i}{eF}\left(\frac{eF}{\sin(eFT)}\mathrm{e}^{-ieFT\dot{G}_{B12}} - \frac{1}{T}\right)\\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - 2\frac{eF}{\sin(eFT)}\mathrm{e}^{-ieFT\dot{G}_{B12}}\,.
\end{aligned}
\tag{3.3}
$$

These are Lorentz matrices, and the above formulas should be understood as power series in the field strength tensor $F_{\mu\nu}$. The scalar function $\dot{G}_{B12}$ is given by $\dot{G}_{B12} = \mathrm{sign}(\tau_1 - \tau_2) - 2(\tau_1-\tau_2)/T$. Due to the translation invariance of those Green's functions one of the integrations $\int d\tau_{a,b}$ is redundant, so that we will set $\tau_b = 0$ in the following.

Removing the second derivative $\ddot{\mathcal{G}}_B$ by a partial integration with respect to τ_a or τ_b one can obtain the equivalent integral

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\mathrm{scal}}(F) = (4\pi)^{-D}\Bigl(-\frac{e^2}{2}\Bigr)&\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar{T}\int_0^T d\tau_a\int_0^T d\tau_b\,\times\\
&\times\det{}^{-\frac{1}{2}}\left[\frac{\sin(eFT)}{eFT}\right]\det{}^{-\frac{1}{2}}\left[\bar{T}-\frac{1}{2}\mathcal{C}_{ab}\right]\times\\
&\times\frac{1}{2}\left\{\mathrm{tr}\,\dot{\mathcal{G}}_{Bab}\,\mathrm{tr}\left[\frac{\dot{\mathcal{G}}_{Bab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\right]+\mathrm{tr}\left[\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb})}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\right]\right\}.
\end{aligned}
\tag{3.4}
$$

We would again like to evaluate this integral for the case of a self-dual field. The worldline correlators (3.3) for such a field simplify to the following [57]:

$$
\begin{aligned}
\mathcal{G}_{B12} &= \frac{T}{2}\left[\frac{1}{Z^2}-\frac{\cosh(Z\dot{G}_{B12})}{Z\sinh(Z)}\right]\mathbb{1} + i\frac{T}{2Z^2}\left[\frac{\sinh(Z\dot{G}_{B12})}{\sinh(Z)} - \dot{G}_{B12}\right]\mathcal{Z}\\
\dot{\mathcal{G}}_{B12} &= \frac{\sinh(Z\dot{G}_{B12})}{\sinh(Z)}\mathbb{1} - i\left[\frac{\cosh(Z\dot{G}_{B12})}{Z\sinh(Z)} - \frac{1}{Z^2}\right]\mathcal{Z}\\
\ddot{\mathcal{G}}_{B12} &= \left[2\delta_{12} - \frac{2}{T}\frac{Z\cosh(Z\dot{G}_{B12})}{\sinh(Z)}\right]\mathbb{1} + i\frac{2}{T}\frac{\sinh(Z\dot{G}_{B12})}{\sinh(Z)}\mathcal{Z}
\end{aligned}
\tag{3.5}
$$

where $\mathcal{Z}_{\mu\nu} \equiv eTF_{\mu\nu}$. The first determinant factor is already known from the one-loop calculation, eq. (2.3). The Lorentz matrix $\mathcal{C}_{ab}$, being an even function of F, for this background becomes scalar:

$$
\begin{aligned}
\mathcal{C}_{ab} &= -C_{ab}\mathbb{1}\,,\\
C_{ab} &= T\frac{\cosh(Z) - \cosh(Z\dot{G}_{Bab})}{Z\sinh(Z)}\,.
\end{aligned}
\tag{3.6}
$$

Therefore in the self-dual case the $\bar{T}$ integrals in eqs. (3.1), (3.4) can be done trivially. In the same way as in the magnetic field case [15, 16], further simplification is achieved by taking the following linear combination of the representations (3.1) and (3.4),

$$
\mathcal{L}^{(2)(SD)}_{\mathrm{scal}}(F) = \frac{D-1}{D}\times \text{eq. (3.1)} + \frac{1}{D}\times\text{eq. (3.4)}\,.
\tag{3.7}
$$

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After the usual rescaling to the unit circle, $\tau_{a,b} = Tu_{a,b}$, and setting $u_b = 0, u_a = u$, we end up with the following twofold integral,[2]

$$\mathcal{L}_{\rm scal}^{(2)(SD)}(f) = \frac{e^2}{2(4\pi)^D}\int_0^\infty \frac{dT}{T}\,{\rm e}^{-m^2T}T^{2-D}\frac{Z^D}{\sinh^2(Z)}2^{\frac{D}{2}}\Big\{I_1(Z,D) + 2\frac{D-1}{D-2}I_2(Z,D)\Big\} \tag{3.8}$$

where

$$\begin{aligned} I_1(Z,D) &= \int_0^1 du \frac{1}{[\cosh(Z)-\cosh(Z(1-2u))]^{\frac{\epsilon}{2}}} \\ I_2(Z,D) &= \int_0^1 du \frac{\cosh(Z(1-2u))}{[\cosh(Z)-\cosh(Z(1-2u))]^{1+\frac{\epsilon}{2}}} \end{aligned} \tag{3.9}$$

($\epsilon = D-4$). We are only interested in terms of order $O(F^4)$ and higher in this lagrangian, since the $O(F^2)$ term contributes only to photon wave function renormalization. For those terms the final T - integration is finite, so that in I_1 we can set $\epsilon = 0$ immediately. The same is not possible for I_2, since this integral for $D = 4$ has a divergence at the points where the two photon end points become coincident, $u = 0, 1$. For the calculation of this integral we split it in the following way,

$$\begin{aligned} I_2(Z,D) = &\int_0^1 du\Big[\cosh(Z(1-2u)) - \cosh\Big(\Big(1-\frac{\epsilon}{2}\Big)Z(1-2u)\Big)\Big]\times \\ &\times\Bigg[\frac{1}{[\cosh(Z)-\cosh(Z(1-2u))]^{1+\frac{\epsilon}{2}}} - \frac{1}{[2Z\sinh(Z)u]^{1+\frac{\epsilon}{2}}} - \\ &- \frac{1}{[2Z\sinh(Z)(1-u)]^{1+\frac{\epsilon}{2}}}\Bigg] + \\ &+\int_0^1 du\,\frac{\cosh(Z(1-2u)) - \cosh\Big((1-\frac{\epsilon}{2})Z(1-2u)\Big)}{[2Z\sinh(Z)u]^{1+\frac{\epsilon}{2}}} + \\ &+\int_0^1 du\,\frac{\cosh(Z(1-2u)) - \cosh\Big((1-\frac{\epsilon}{2})Z(1-2u)\Big)}{[2Z\sinh(Z)(1-u)]^{1+\frac{\epsilon}{2}}} + \\ &+\int_0^1 du\,\frac{\cosh\Big((1-\frac{\epsilon}{2})Z(1-2u)\Big)}{[\cosh(Z)-\cosh(Z(1-2u))]^{1+\frac{\epsilon}{2}}}\,. \end{aligned} \tag{3.10}$$

Of these four integrals the first one is of order O(ϵ) and can be omitted. The second and third ones are equal and elementary,

$$\int_0^1 du\,\frac{\cosh(Z(1-2u)) - \cosh\Big((1-\frac{\epsilon}{2})Z(1-2u)\Big)}{[2Z\sinh(Z)u]^{1+\frac{\epsilon}{2}}} = -\frac{1}{2} + O(\epsilon)\,. \tag{3.11}$$

The fourth integral can be calculated by transforming to the variable $\phi = (1-2u)Z$, and using formula 22.10.11 of [58] (see [59] for a general technique for computing this type of

[2]Note that the δ_{ab}-term contained in $\ddot{\mathcal{G}}_{Bab}$ has been deleted, since here the photon proper-time integral is a tadpole type integral which vanishes in dimensional regularization.

integral). The result is

$$\frac{1}{Z}\int_0^Z d\phi\,\frac{\cosh((1-\frac{\epsilon}{2})\phi)}{[\cosh(Z)-\cosh(\phi)]^{1+\frac{\epsilon}{2}}} = 2^{-1-\frac{\epsilon}{2}}B\left(-\frac{\epsilon}{2},-\frac{\epsilon}{2}\right)\frac{\cosh(Z)}{Z\Bigl(\sinh(Z)\Bigr)^{1+\epsilon}} \tag{3.12}$$

with B the Euler Beta function. Expanding in ϵ we find

$$\begin{aligned}\mathcal{L}_{\rm scal}^{(2)(SD)}(f_0) = \frac{e_0^2}{(4\pi)^4}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m_0^2T}\Biggl\{\cosh(Z)\left(\frac{Z}{\sinh(Z)}\right)^3\Biggl[-\frac{12}{\epsilon}+2+12\ln(4\pi T)-\\ -12\ln\left(\frac{Z}{\sinh(Z)}\right)\Biggr]-4\frac{Z^4}{\sinh^2(Z)}\Biggr\}\end{aligned} \tag{3.13}$$

We have written this result for the lagrangian in terms of bare quantities, since it still requires renormalization. Mass renormalization requires us to subtract a term $\delta m_0^2\frac{\partial}{\partial m_0^2}\mathcal{L}_{\rm scal}^{(1)(SD)}(f_0)$, where $\mathcal{L}_{\rm scal}^{(1)(SD)}(f_0)$ is the one-loop Euler-Heisenberg lagrangian (2.4), and δm_0^2 the one-loop mass shift, both calculated in dimensional regularization. The mass shift is

$$\delta m_0^2 = m_0^2\frac{\alpha_0}{4\pi}\left[-\frac{6}{\epsilon}+7-3[\gamma-\ln(4\pi)]-3\ln(m_0^2)\right]+O(\epsilon) \tag{3.14}$$

Noting that

$$\frac{\cosh(Z)}{\sinh^3(Z)} = -\frac{1}{2}\frac{d}{dZ}\frac{1}{\sinh^2(Z)} \tag{3.15}$$

we perform a partial integration on the first term on the right hand side of (3.13). After this the pole term in it takes the same form as the pole of the mass shift term above. Subtracting the mass shift term we end up with the following expression for the renormalized lagrangian,

$$\mathcal{L}_{\rm scal}^{(2)(SD)}(f) = \frac{e^2}{(4\pi)^4}\int_0^\infty \frac{dT}{T^3}\mathrm{e}^{-m^2T}\left(\frac{Z}{\sinh(Z)}\right)^2\Biggl\{-4Z^2+3m^2T\Biggl[1-\gamma+\ln\left(\frac{Z}{\sinh(Z)m^2T}\right)\Biggr]\Biggr\} \tag{3.16}$$

This can also be written as

$$\begin{aligned}\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) = m^4\frac{e^2}{(4\pi)^4}\frac{1}{\kappa^2}\int_0^\infty \frac{dZ}{Z^3}\,\mathrm{e}^{-2\kappa Z}\left(\frac{Z}{\sinh(Z)}\right)^2\times\\ \times\Biggl\{-Z^2+\frac{3}{2}\kappa Z\Biggl[1-\gamma+\ln\left(\frac{Z}{\sinh(Z)}\right)-\ln(2\kappa Z)\Biggr]\Biggr\}.\end{aligned} \tag{3.17}$$

Note that, since we have not yet performed the photon wave function renormalization, this formula is correct only beginning from $O(1/\kappa^4)$.

For the calculation of this integral, we go back to the regularized version, eqs. (3.8)–(3.12), and note that it involves essentially only the integral

$$\int_0^\infty dx\,\mathrm{e}^{-2\kappa x}\sinh^\gamma(x) = 2^{-\gamma-1}B\Bigl(\gamma+1,\kappa-\frac{\gamma}{2}\Bigr). \tag{3.18}$$

However, to be able to take the limit $\epsilon\to 0$ we instead introduce

$$I(\kappa,\epsilon) \equiv \int_0^\infty dx\,\mathrm{e}^{-2\kappa x}\left[\frac{1}{\sinh^{2+\epsilon}(x)}-\frac{1}{x^{2+\epsilon}}\right]. \tag{3.19}$$

Including the mass renormalization term, and again disregarding $O(f^2)$ terms, we can then write

$$\mathcal{L}^{(2)(SD)}_{\rm scal}(f_0)-\delta m_0^2\frac{\partial}{\partial m_0^2}\mathcal{L}^{(1)(SD)}_{\rm scal}(f_0) = m^4\frac{e^2}{(4\pi)^4}\frac{1}{2\kappa}\biggl\{3(1-\gamma-\ln(2\kappa))+\frac{1}{\kappa}\frac{\partial}{\partial\kappa}+3\frac{\partial}{\partial\epsilon}\biggr\}I(\kappa,\epsilon)|_{\epsilon=0}. \tag{3.20}$$

Using (3.18) this yields, after an easy computation, our final result (1.4),

$$\mathcal{L}^{(2)(SD)}_{\rm scal}(\kappa) = \alpha\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\xi^2(\kappa)-\xi'(\kappa)\right]$$

where we have finally subtracted out the terms of order $O(f^0)$, $O(f^2)$.

3.2 Spinor QED

The calculations for the spinor QED case are very similar, so we content ourselves primarily with a presentation of the results.

For the two-loop Euler-Heisenberg lagrangian in euclidean spinor QED, the worldline formalism leads to the following parameter integral, which is completely analogous to (3.4) [15, 16]:

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm spin}(F) &= (4\pi)^{-D}e^2\int_0^\infty\frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar{T}\int_0^T d\tau_a\int_0^T d\tau_b\times\\ &\quad\times\det{}^{-\frac{1}{2}}\left[\frac{\tan(eFT)}{eFT}\left(\bar{T}-\frac{1}{2}\mathcal{C}_{ab}\right)\right]\times\\ &\quad\times\frac{1}{2}\biggl\{\mathrm{tr}\,\dot{\mathcal{G}}_{Bab}\,\mathrm{tr}\left[\frac{\dot{\mathcal{G}}_{Bab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\right]-\mathrm{tr}\,\mathcal{G}_{Fab}\,\mathrm{tr}\left[\frac{\mathcal{G}_{Fab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\right]+\\ &\qquad+\mathrm{tr}\left[\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb}+\mathcal{G}_{Faa})+\mathcal{G}_{Fab}\mathcal{G}_{Fab}-\mathcal{G}_{Faa}\mathcal{G}_{Fbb}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\right]\biggr\}.\end{aligned} \tag{3.21}$$

The bosonic worldline Green's function $\mathcal{G}_B$ is given in (3.3), while the fermionic one is:

$$\mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\,\frac{e^{-ieFT\dot{G}_{B12}}}{\cos(eFT)} \tag{3.22}$$

where $G_{F12} = \mathrm{sign}(\tau_1-\tau_2)$. For a self-dual background $\mathcal{G}_F$ simplifies to

$$\mathcal{G}_{F12} = G_{F12}\frac{\cosh(efT\dot{G}_{B12})}{\cosh(efT)}\mathbb{1}-iG_{F12}\frac{\sinh(efT\dot{G}_{B12})}{efT\,\cosh(efT)}\,\mathcal{Z}\,. \tag{3.23}$$

The calculation, together with the charge and mass renormalization, proceed in a manner analogous to the two-loop scalar QED calculation in the previous section (for details of the mass renormalization, see [15]). After similar manipulations we find that the on-shell renormalized effective lagrangian can be written as the following proper-time integral:

$$\begin{aligned}\mathcal{L}^{(2)(SD)}_{\rm spin} &= -\alpha\frac{m^4}{16\pi^3}\frac{2}{\kappa}\int_0^\infty dZ e^{-2\kappa Z}\biggl\{\left(-Z+\coth Z\log\frac{\sinh Z}{Z}\right)\left(1-\frac{1}{4\kappa Z}\frac{d}{dZ}\right)\coth Z+\\ &\qquad+\left(\log(2\kappa Z)+\gamma-\frac{5}{6}\right)\left(1-\frac{1}{8\kappa Z}\frac{d}{dZ}\right)\times\\ &\qquad\times\left(\coth^2 Z-\frac{1}{Z^2}-\frac{2}{3}\right)+\frac{5}{6}+\frac{5}{24\kappa Z}\biggr\}.\end{aligned} \tag{3.24}$$

Here we have also subtracted the zero-field term and done the charge renormalization.

As in the scalar case, this proper-time integral can be done in closed-form. After manipulations similar to those in the scalar case, we find the final result (1.5),

$$\mathcal{L}_{\rm spin}^{(2)(SD)}(\kappa) = -\alpha\,\frac{m^4}{(4\pi)^3}\frac{2}{\kappa^2}\left[3\xi^2(\kappa) - \xi'(\kappa)\right]$$

where $\xi(\kappa)$ is the same function that was defined before in (1.7). This result (1.5) is remarkably similar to the two-loop scalar QED result (1.4). We find it interesting that the spinor and scalar QED results for a constant selfdual background can each be written in such a simple form in terms of the same function $\xi(\kappa)$. This fact is discussed in the next section.

4. Self-duality, helicity and supersymmetry at one- and two-loops

In this section we explain why the renormalized scalar and spinor effective lagrangians in a self-dual background are so similar, at both one- and two-loop. The self-duality of the background has the effect that both the scalar and spinor cases are dramatically simplified from the results for a general constant background. But, on top of these simplifications, the scalar and spinor results (1.4) and (1.5) are remarkably similar to one another, each being expressed as a simple function of $\xi(\kappa)$. The key to understanding this correspondence is the connection between self-duality, helicity and supersymmetry. It is well known that self-dual gauge fields are closely related to supersymmetry, and that they correspond to helicity eigenstates [30, 34, 20, 35]. In this section we explain how these connections are manifest in our exact effective action results at both one- and two-loops.

Consider the one-loop QED case. Since $\coth^2 t = 1 + 1/\sinh^2 t$, we compare the one-loop renormalized effective lagrangians (2.4) and (2.9) for the scalar and spinor cases, respectively, and find that

$$\mathcal{L}_{\rm spin}^{(1)(SD)}(\kappa) = -2\,\mathcal{L}_{\rm scal}^{(1)(SD)}(\kappa) \tag{4.1}$$

Note that this proportionality between the scalar and spinor effective lagrangians is only true for a self-dual background. Furthermore, the relation (4.1) holds for the renormalized effective lagrangians, not for the unrenormalized ones.

This proportionality between the one-loop scalar and spinor effective lagrangians reflects a well-known supersymmetry of the self-dual background [30, 34, 20, 35]. The basic relation between self-dual fields and helicity can be traced to the following identity (we work in euclidean space):

$$(\gamma_\mu D_\mu)^2\left(\frac{1+\gamma_5}{2}\right) = (D_\mu D_\mu)\left(\frac{1+\gamma_5}{2}\right) \tag{4.2}$$

which holds provided the field strength $F_{\mu\nu}$ is self-dual. This fact has the consequence [30, 20] that the bosonic and fermionic operators

$$\Delta_B = -D_\mu D_\mu + m^2 \tag{4.3}$$

$$\Delta_F = -\left(\gamma_\mu D_\mu\right)^2 + m^2 = -D_\mu D_\mu - \frac{ie}{2}\sigma_{\mu\nu}F_{\mu\nu} + m^2 \tag{4.4}$$

have identical spectra, but the fermionic case has a four-fold multiplicity (from the spinor degrees of freedom). This is due to a quantum mechanical supersymmetry [34] relating the bosonic and fermionic operators in (4.3) and (4.4). Therefore,

$$\mathcal{L}_{\rm spin}^{(1)(SD)} = \frac{1}{2}\log\det\Delta_F = 2\log\det\Delta_B = -2\mathcal{L}_{\rm scal}^{(1)(SD)} \tag{4.5}$$

which is precisely the one-loop relation (4.1).

Another equivalent way to see this is to note that because of the helicity/self-duality relation (4.2), the spinor propagator in a self-dual background can be expressed in a simple manner in terms of the scalar propagator in the same background [35]:

$$S = -\left(\not{D} - m\right) G \left(\frac{1+\gamma_5}{2}\right) - G\,\not{D}\left(\frac{1-\gamma_5}{2}\right) + \frac{1}{m}\left(1 + \not{D}G\not{D}\right)\left(\frac{1-\gamma_5}{2}\right) \tag{4.6}$$

where the scalar and spinor propagators are

$$G = \frac{1}{-D^2 + m^2}\,, \qquad S = \frac{1}{\not{D} + m}\,. \tag{4.7}$$

This relation between the spinor and scalar propagators in a self-dual background provides another perspective on the relation (4.1). The spinor induced current can be related to the scalar induced current since,

$$\begin{aligned}\langle J^\mu_{\rm spin}\rangle &= \frac{\delta}{\delta A_\mu}\,\mathcal{L}^{(1)}_{\rm spin} \\ &= -ie\,{\rm tr}\langle\gamma^\mu\, S\rangle \\ &= 2ie\langle D_\mu G + G D_\mu\rangle\end{aligned} \tag{4.8}$$

where in the last step we have used (4.6). This current relation agrees with the one-loop effective lagrangian relation (4.1).

At the two-loop level, the supersymmetry relation (4.1) no longer holds, as can be seen by comparing the two-loop scalar and spinor results (1.4) and (1.5), respectively. Namely,

$$\mathcal{L}_{\rm spin}^{(2)(SD)}(\kappa) + 2\mathcal{L}_{\rm scal}^{(2)(SD)}(\kappa) = -3m^4\frac{\alpha}{(4\pi)^3}\,\frac{\xi^2(\kappa)}{\kappa^2}\,. \tag{4.9}$$

This non-vanishing is because the two-loop effective lagrangians are not simply logarithms of determinants. Nevertheless, it is interesting to see that each of the scalar and spinor two-loop effective lagrangians can be written in such a simple form in terms of the same function $\xi(\kappa)$. To understand why this is the case, let us reconsider this computation in terms of more standard field theory techniques. Using Ritus's approach [12] the two-loop effective lagrangians can be written as [12]:

$$\mathcal{L}^{(2)}_{\rm spin} = \frac{e^2}{2}\int dx'\mathcal{D}(x-x')\,{\rm tr}\left[\gamma_\mu S(x,x')\gamma_\mu S(x',x)\right] \tag{4.10}$$

$$\mathcal{L}^{(2)}_{\rm scal} = -e^2\int dx'\mathcal{D}(x-x')\left[D_\mu G(x,x')\,D_\mu G(x',x) + D_\mu G(x,x')\,\overleftarrow{D}_\mu\,G(x',x)\right] \tag{4.11}$$

where $S(x,x')$ and $G(x,x')$ are the position space spinor and scalar propagators in the presence of the background field, and $\mathcal{D}(x,x')$ denotes the internal photon propagator. We have not written tadpole terms, as these do not contribute after renormalization. These expressions (4.10) and (4.11) are valid for a general background. Now, if we specialize to a self-dual background, then the spinor and scalar propagators become related as in (4.6), which means that the Dirac traces can be done in the spinor case, leading to an expression in terms of the scalar propagator:

$$\mathcal{L}^{(2)}_{\rm spin} = \frac{e^2}{2}\int dx'\mathcal{D}(x-x')\left[-8D_\mu G(x,x')\,D_\mu G(x',x) + 16 D_\mu G(x,x')\,\overleftarrow{D}_\mu\,G(x',x)\right]. \tag{4.12}$$

Thus, comparing with the scalar two-loop effective lagrangian (4.11), we see that the spinor two-loop effective lagrangian for a self-dual background involves the same structures but with different coefficients. This is reflected directly in our explicit closed-form two-loop expressions (1.4) and (1.5).

To make this connection more precise, we consider a general linear combination of these two structures:

$$\int dx'\mathcal{D}(x-x')\left[A\,D_\mu G(x,x')\,D_\mu G(x',x) + B\,D_\mu G(x,x')\,\overleftarrow{D}_\mu\,G(x',x)\right] \tag{4.13}$$

where A and B are numerical coefficients. Now, in a constant self-dual background, the position space scalar propagator has the following simple form (up to a phase that is not important here)

$$G(x,x') = \frac{ef}{(4\pi)^2}\int_0^\infty \frac{dt}{\sinh^2(t)}\exp\left[-2\kappa t - \frac{ef}{4}\coth(t)(x-x')^2\right] \tag{4.14}$$

This means that we can express the general linear combination in (4.13) as

$$\begin{aligned}
&\frac{m^4}{(4\pi)^4\kappa^2}\int_0^\infty dt\int_0^\infty dt'\frac{e^{-2\kappa(t+t')}}{\sinh^2(t)\sinh^2(t')}\left\{\frac{\frac{1}{2}(A+B)}{(\coth(t)+\coth(t'))^2} - \frac{\frac{A}{4}\coth(t+t')-\frac{B}{2}\kappa}{(\coth(t)+\coth(t'))}\right\} = \\
&= \frac{m^4}{(4\pi)^4\kappa^2}\int_0^\infty ZdZe^{-2\kappa Z}\int_0^1 du\left\{\frac{\frac{1}{2}(A+B)}{\sinh^2(Z)} - \frac{\frac{A}{2}\coth(Z)-B\kappa}{\sinh(Z)(\cosh(Z)-\cosh(Z(1-2u)))}\right\}
\end{aligned} \tag{4.15}$$

where in the last step we have defined new integration variables $Z = t+t'$, and $u = \frac{t'}{t+t'}$. After renormalization, and using similar integration techniques as in section 3, the first term inside the curly parentheses in (4.15) leads to ξ', with a numerical coefficient $\frac{1}{2}(A+B)$, while the second term leads to ξ^2, with a coefficient $-(\frac{A}{2}+B)$. Thus, the general linear combination in (4.13) leads, after renormalization, to

$$\begin{aligned}
&\int dx'\mathcal{D}(x-x')\left[A\,D_\mu G(x,x')\,D_\mu G(x',x) + B\,D_\mu G(x,x')\,\overleftarrow{D}_\mu\,G(x',x)\right] = \\
&\qquad = \frac{m^4}{(4\pi)^4}\frac{1}{\kappa^2}\left[-\left(\frac{A}{2}+B\right)\xi^2(\kappa) + \frac{1}{2}(A+B)\xi'(\kappa)\right]
\end{aligned} \tag{4.16}$$

Now, the two-loop scalar QED effective lagrangian (4.11) has numerical coefficients $A = B = 1$, and an overall factor of $-e^2$, which then leads immediately to our scalar result (1.4).

On the other hand, the two-loop spinor QED effective lagrangian (4.12) has $A = -8$ and $B = 16$, and an overall factor of $e^2/2$, which then leads immediately to our spinor result (1.5). Thus, the simple relation between the two-loop spinor and scalar effective lagrangians for a self-dual background is ultimately due to the relation (4.6) between the spinor and scalar propagators in a self-dual background.

5. The all '+' helicity amplitudes in the Euler-Heisenberg approximation

As is well known, the Euler-Heisenberg lagrangian contains the information about the low-energy limit of the N-photon amplitudes at the given loop level. After specialization to a self-dual field it still contains information about the component of this amplitude which has all helicities alike. This component of the N-photon or gluon amplitudes generally leads to especially compact expressions [36]–[42][60]. In this section, we will apply our results for the self-dual Euler-Heisenberg lagrangians to an explicit computation of these amplitudes in the low-energy limit, at one and two loops.

At the one loop level it is not difficult to perform a direct calculation of these amplitudes in the Bern-Kosower formalism. We present this calculation, too, since it is quite instructive to see how the equivalence between both approaches comes about.

The Bern-Kosower master formula [45], together with a judiciously chosen set of partial integrations [45, 62, 63], allows one to write the one-loop N-photon amplitude in scalar QED in the following form:[3]

$$\Gamma^{(1)}_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] = -\frac{e^N}{16\pi^2}\int_0^\infty \frac{ds}{s}e^{-im^2s}s^{N-2}\int_0^1 du_1\int_0^1 du_2\cdots\int_0^1 du_N \times \\ \times Q_N(\dot G_{Bij})\exp\left\{-is\sum_{i<j=1}^N G_{Bij}k_i\cdot k_j\right\} \qquad (5.1)$$

Here $G_{Bij} =\mid u_i - u_j \mid -(u_i-u_j)^2$, $\dot G_{Bij} = {\rm sign}(u_i-u_j) - 2(u_i-u_j)$. This representation holds on- and off-shell. It is important to note that the right-hand side represents the complete amplitude, not just a particular Feynman diagram. See [61, 53] for the relation of this representation to standard Feynman parameter integrals. The exponential factor $\exp\{-is\sum_{i<j=1}^N G_{Bij}k_i\cdot k_j\}$ corresponds to the standard one-loop N - point denominator, while Q_N corresponds to a numerator polynomial. This polynomial depends on the various $\dot G_{Bij}$'s as well as on the kinematic invariants $\varepsilon_i\cdot k_j$, $\varepsilon_i\cdot\varepsilon_j$. Q_N can be decomposed in terms of 'cycles' and 'tails' as follows:[4] define the 'Lorentz cycle' Z_n by

$$\begin{aligned} Z_2(ij) &\equiv \frac{1}{2}\,{\rm tr}(F_iF_j) \\ Z_n(i_1i_2\dots i_n) &\equiv {\rm tr}(F_{i_1}F_{i_2}\cdots F_{i_n}) \qquad (n\geq 3) \end{aligned} \qquad (5.2)$$

where $(F_i)_{\mu\nu} \equiv k_{i\mu}\varepsilon_{i\nu} - \varepsilon_{i\mu}k_{i\nu}$ is the field strength tensor for leg 'i'. Define the 'bi-cycle of length n', $\dot G(i_1i_2\dots i_n)$, by

$$\dot G(i_1i_2\dots i_n) \equiv \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_{n-1}i_n}\dot G_{Bi_ni_1}Z_n(i_1i_2\dots i_n)\,. \qquad (5.3)$$

[3] In this section we use Minkowski space $(+---)$ conventions.

[4] We follow the notation of [63] rather than [53].

Then Q_N has a decomposition of the form

$$Q_N = \sum Q_N^{z_1 z_2 \cdots z_i} \tag{5.4}$$

where the superscripts on a term indicate that it contains i bi-cycle factors of lengths $z_1, \ldots, z_i$. Unless all indices $1, \ldots, N$ in a given term are bound up in cycle factors there will be one more factor, called 'tail'.

One advantage of this representation is that the polynomial Q_N is homogeneous in the momenta; each term in it has exactly N factors of momentum. Therefore the low-energy limit of the amplitude, corresponding to the case where m^2 becomes much larger than all kinematic invariants $k_i \cdot k_j$, is simply obtained by replacing the exponential factor $\exp\{-is \sum_{i<j=1}^N G_{Bij} k_i \cdot k_j\}$ by unity. The omission of this factor not only decouples the global proper-time integral, but also turns all terms involving tails into total derivatives. This leaves us with only those numerator terms which are products of bi-cycles. Thus the low energy, or Euler-Heisenberg ('EH'), limit of this amplitude can be written as

$$\Gamma_{\text{scal}}^{(1)(EH)}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = \frac{(-ie)^N \Gamma(N-2)}{16\pi^2 m^{2N-4}} \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N \, Q_N^{(EH)}(\dot{G}_{Bij}) \tag{5.5}$$

where

$$Q_N^{(EH)}(\dot{G}_{Bij}) = \sum_{\text{part.}} P(\{i_1 i_2 \ldots i_{k_1}\}) P(\{i_{k_1+1} \ldots i_{k_1+k_2}\}) \cdots P(\{i_{k_1+k_2+\ldots k_{j-1}+1} \ldots i_N\}) \,. \tag{5.6}$$

Here the sum is over all possible partitions of the set of indices $1, \ldots, N$ into subsets, and $P(\{i_1 i_2 \ldots i_k\})$ denotes the sum over all *distinct* bi-cycles which can be formed with the given subset of indices, e.g. $P(\{ijkl\}) = \dot{G}(ijkl) + \dot{G}(ijlk) + \dot{G}(ikjl)$. The integral $\int_0^1 du_1 \cdots du_N$ factorizes into copies of the basic 'chain integral' [47]

$$b_n \equiv \int_0^1 du_1 du_2 \ldots du_n \, \dot{G}_{B12} \dot{G}_{B23} \cdots \dot{G}_{Bn1} = \begin{cases} -2^n \frac{\mathcal{B}_n}{n!} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{5.7}$$

where $\mathcal{B}_n$ denotes the nth Bernoulli number. The result of the integrations can therefore be written as

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_N \, Q_N^{(EH)}(\dot{G}_{Bij}) = \tag{5.8}$$
$$= \sum_{\text{part.}} b_{k_1} F_{k_1}(\{i_1 i_2 \ldots i_{k_1}\}) b_{k_2} F_{k_2}(\{i_{k_1+1} \ldots i_{k_1+k_2}\}) \cdots b_{k_j} F_{k_j}(\{i_{k_1+k_2+\ldots k_{j-1}+1} \ldots i_N\})$$

where $F_k(\{i_1 i_2 \ldots i_k\})$ denotes the sum over all distinct Lorentz cycles which can be formed with a given subset of indices, e.g. $F_4(\{ijkl\}) = Z_4(ijkl) + Z_4(ijlk) + Z_4(ikjl)$. This allows us to write the amplitude in the following form:

$$\Gamma_{\text{scal}}^{(1)(EH)}[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = \frac{(-ie)^N (N-1)!}{16\pi^2 m^{2N-4}} \times$$
$$\times \exp\left\{ \sum_{m=1}^{\infty} b_{2m} \sum_{\{i_1 \ldots i_{2m}\}} F_{2m}(\{i_1 i_2 \ldots i_{2m}\}) \right\} \Bigg|_{(1 \ldots N)} \,. \tag{5.9}$$

Here the second sum in the exponent runs over the set of all sets of $2m$ different positive numbers, and it is understood that after expansion of the exponential only those terms are kept which contain each index $1, 2, \dots, N$ precisely once.

It takes only a minor modification to generalize this computation to the spinor loop case: according to the Bern-Kosower 'replacement' rules, the transition from scalar to spinor QED can be effected by replacing the above 'chain integral' (5.7) by the 'super chain integral'

$$b_n - f_n \equiv \int_0^1 du_1 du_2 \dots du_n \left(\dot{G}_{B12}\dot{G}_{B23}\cdots\dot{G}_{Bn1} - G_{F12}G_{F23}\cdots G_{Fn1}\right) = (2 - 2^n)\, b_n \,. \tag{5.10}$$

As usual one must also multiply by a global factor of (-2) for statistics and degrees of freedom.

Let us now compare with the calculation of the same amplitude via the effective action. In (1.2) we wrote the one-loop Euler-Heisenberg lagrangian in terms of the two Maxwell invariants. Alternatively, the integrand can also be written in terms of traces of powers of the field strength tensor [15]:

$$\mathcal{L}^{(1)}_{\rm scal}(F) = -\frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^3}\, \mathrm{e}^{-ism^2} \exp\left\{\frac{1}{2}\sum_{m=1}^\infty \frac{b_{2m}}{2m}(es)^{2m}\, \mathrm{tr}[F^{2m}]\right\}. \tag{5.11}$$

According to standard field theory (see, e.g., [64]) the low energy limit of the N-photon amplitude can be obtained from the effective lagrangian by replacing F with $i\sum_{i=1}^N F_i$, expanding out, and collecting all those terms which involve each $F_1, \dots, F_N$, precisely once. The equivalence of (5.9) and (5.11) therefore follows immediately from the combinatorial fact that

$$\mathrm{tr}\Big[(F_1 + \cdots + F_N)^n\Big]\Big|_{\text{all different}} = 2n \sum_{\{i_1 \dots i_n\}} F_n(\{i_1 i_2 \dots i_n\}) \qquad (2 \le n \le N)\,. \tag{5.12}$$

Here the sum runs over all possiblities of choosing a subset of n indices out of $\{1, \dots, N\}$. On the left hand side it is understood that terms containing the same F_i twice are to be discarded.

To obtain the on-shell helicity amplitudes, one now needs to construct polarisation vectors for all possible inequivalent helicity assignments, and compute the various $\varepsilon_i \cdot k_j$'s and $\varepsilon_i \cdot \varepsilon_j$'s. For small numbers of photons, this could be easily worked out for arbitrary assignments, using standard techniques such as the spinor helicity formalism [65, 66, 67, 68, 36]. In this formalism, a polarisation vector with circular polarisation '$\pm$' for a photon with momentum k is written as

$$\varepsilon_\mu^\pm = \pm \frac{\langle q^\mp \mid \gamma_\mu \mid k^\mp\rangle}{\sqrt{2}\langle q^\mp \mid k^\pm\rangle}\,. \tag{5.13}$$

Here $\langle q^\pm \mid k^\mp\rangle = \overline{u_\pm(q)}u_\mp(k)$ etc. are basic spinor products, and q is a reference momentum which is different from k but arbitrary otherwise; see [68] for details and conventions. We will be concerned here only with the all '+' amplitudes (see, however, the last remark at the end of this section). For those, we know that all field strength tensors $F_1, \ldots, F_N$, and therefore also their sum $F_{\rm tot} \equiv \sum_{i=1}^N F_i$, are self-dual tensors. Therefore, as in (2.2) the square of $F_{\rm tot}$ must be proportional to the Lorentz identity. And indeed, using the spinor helicity formalism it is easy to see that

$$\begin{aligned} F_i^2 &= 0 \\ (F_iF_j + F_jF_i)^{\mu\nu} &= -\frac{1}{2}[ij]^2\eta^{\mu\nu} \end{aligned} \tag{5.14}$$

where $[i\,j] \equiv \langle k_i{}^+ \mid k_j{}^-\rangle$. It follows that

$$\begin{aligned} F_{\rm tot}^2 &= -f_{\rm tot}^2\mathbb{1} \\ f_{\rm tot}^2 &\equiv \frac{1}{2}\sum_{1\le i<j\le N}[ij]^2 \end{aligned} \tag{5.15}$$

and

$$\mathrm{tr}[F_{\rm tot}^{2m}] = 4(-1)^m\left(f_{\rm tot}^2\right)^m . \tag{5.16}$$

Thus, the all '+' component of the N-point amplitude can be obtained from the self-dual effective lagrangian (2.4) as follows: expand this lagrangian in powers of f, i.e. in the weak field expansion; explicitly,

$$\begin{aligned} \mathcal{L}_{\rm scal}^{(1)(SD)}(f) &= \frac{m^4}{(4\pi)^2}\sum_{n=2}^\infty c_{\rm scal}^{(1)}(n)\frac{(2ef)^{2n}}{m^{4n}} \\ c_{\rm scal}^{(1)}(n) &= -\frac{\mathcal{B}_{2n}}{2n(2n-2)} . \end{aligned} \tag{5.17}$$

The $N = 2n$ point amplitude is then obtained simply by taking the term involving f^N, replacing f^2 by $f_{\rm tot}^2$, and keeping only those terms in the expansion of $f_{\rm tot}^N$ involving each index precisely once:

$$\begin{aligned} \Gamma_{\rm scal}^{(1)(EH)}[k_1,\varepsilon_1^+;\ldots;k_N,\varepsilon_N^+] &= \frac{(2e)^N}{(4\pi)^2 m^{2N-4}}\,c_{\rm scal}^{(1)}(\tfrac{N}{2})\chi_N \\ \chi_N &\equiv f_{\rm tot}^N\Big|_{\text{all different}} \\ &= \frac{(\frac{N}{2})!}{2^{\frac{N}{2}}}\Big\{[12]^2[34]^2\cdots[(N-1)N]^2 + \text{ all permutations}\Big\} \end{aligned} \tag{5.18}$$

To generalize this result to the spinor QED and two-loop cases, all we need are the corresponding weak field expansions. For the one-loop amplitudes in spinor QED, we infer from (4.1) that they are proportional to the scalar QED ones:

$$\Gamma_{\rm spin}^{(1)(EH)}[k_1,\varepsilon_1^+;\ldots;k_N,\varepsilon_N^+] = -2\Gamma_{\rm scal}^{(1)(EH)}[k_1,\varepsilon_1^+;\ldots;k_N,\varepsilon_N^+] . \tag{5.19}$$

At the two-loop level, we can easily obtain the weak field expansions of the lagrangians (1.4), (1.5) from the well-known asymptotic expansion of the digamma function at infinity (see part 2). This yields

$$\Gamma^{(2)(EH)}_{\rm scal}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+] = \alpha\pi\frac{(2e)^N}{(4\pi)^2 m^{2N-4}}\,c^{(2)}_{\rm scal}(\tfrac{N}{2})\chi_N$$

$$c^{(2)}_{\rm scal}(n) = \frac{1}{(2\pi)^2}\biggl\{\frac{2n-3}{2n-2}\,\mathcal{B}_{2n-2}+\frac{3}{2}\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\biggr\} \tag{5.20}$$

for scalar QED, and

$$\Gamma^{(2)(EH)}_{\rm spin}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+] = -2\alpha\pi\frac{(2e)^N}{(4\pi)^2 m^{2N-4}}\,c^{(2)}_{\rm spin}(\tfrac{N}{2})\chi_N$$

$$c^{(2)}_{\rm spin}(n) = \frac{1}{(2\pi)^2}\biggl\{\frac{2n-3}{2n-2}\,\mathcal{B}_{2n-2}+3\sum_{k=1}^{n-1}\frac{\mathcal{B}_{2k}}{2k}\frac{\mathcal{B}_{2n-2k}}{(2n-2k)}\biggr\} \tag{5.21}$$

for spinor QED.

Let us discuss some implications of these results. In the massless case the one-loop all '+' N-photon amplitudes have been shown to vanish for $N \geq 6$ [37]. Our result shows that this miracle does not generalize to the massive case at either one or two loops.

The one-loop relation (5.19) is a direct consequence of the supersymmetry Ward identities for S-matrix elements first obtained by Grisaru, Pendleton and van Nieuwenhuizen [69, 70]. Those identities hold for massive as well as for massless supermultiplets in the loop. As was already clear from (4.9) this relation (5.19) does not generalize to the two-loop level. The reason is, of course, that at two loops ordinary QED 'knows' already that it is not supersymmetric. The corresponding supersymmetry identity at the two-loop level involves, in addition to the scalar and fermion loop pieces (5.20), (5.21), contributions from diagrams with a photino exchange as well as diagrams containing the scalar quartic coupling. At two loops our results could still be used together with the supersymmetry Ward identity to infer the sum of those additional contributions to the 'all +' amplitude in the Euler-Heisenberg approximation. See the recent [39] for an explicit verification of the vanishing of the two — loop four — photon 'all +' amplitude in massless $N = 1$ and $N = 2$ SUSY QED.

Now that we have the above explicit all - N formulas, it would be of obvious interest to study the large-N behaviour of the coefficients $c^{(1,2)}_{\rm scal,spin}(\frac{N}{2})$. This will be done in detail in part 2. For now, let us just anticipate the facts that the leading asymptotic growth of the coefficients turns out to be factorial, and to be the same at one and two loops.

Finally, let us mention that the same method of computation applied to the photon amplitudes with all but one helicity alike $(-++\dots+)$ allows us to show without computation that those vanish in the low energy limit. Namely, gauge invariance implies that the Euler-Heisenberg lagrangian can, at arbitrary loop order and at an arbitrary fixed order in the weak field expansion, be written as a linear combination of terms involving only products of ${\rm tr}[F^n]$'s. After substituting for F the sum of the individual F_i's this yields sums of terms involving only products of Z_n's, where one of the factors must contain the F_i which has negative helicity, say, F_1. But using the spinor helicity formalism it is easy to show that a ${\rm tr}(F_1^- F_{i_1}^+ \cdots F_{i_k}^+)$ always vanishes (use k_1 as the reference momentum for all other legs). Therefore the Euler-Heisenberg limits of these amplitudes must vanish to all orders in perturbation theory. This argument applies, of course, to both scalar and spinor QED.

6. Conclusions

To summarize, we have computed the on-shell renormalized two-loop effective lagrangian for both scalar and spinor QED, for the case of a constant self-dual euclidean background. This case exhibits remarkable simplifications compared to the general constant background, and even compared to the constant magnetic (or electric) background. For the self-dual case the propagators simplify to such an extent that the final renormalized answers can be given in a simple closed form in terms of a common function $\xi(\kappa)$, which is essentially the digamma function. Although for ϕ^4 theory an analogous result has been known for a long time [71] we are not aware of any comparable two-loop result in gauge theory.

Furthermore, the scalar and spinor QED cases are very similar to one another, a fact that we explained using the relation between self-duality, supersymmetry and helicity, which has the implication that the spinor propagator in a self-dual background has a simple expression in terms of the scalar propagator in the same background. At one-loop this implies that the scalar and spinor effective lagrangians are proportional, but at two-loop the relation is more involved. This provides a new manifestation of the well-known simplicities arising in helicity (or self-duality) projections.

Our explicit results for the self-dual two-loop effective lagrangians have allowed us to compute the low energy limit of the massive N-photon amplitudes with all helicities alike, at one and two loops, for both scalar and spinor QED. This shows, in particular, that the vanishing theorem mentioned above does not extend to massive amplitudes. As a side result, we have shown that the amplitudes with all but one helicity alike vanish in the same approximation, to all orders in perturbation theory.

From the mathematical point of view, we find it intriguing that, for both scalar and spinor QED and at both one and two loops, the final results for the effective lagrangian turned out to be expressible in terms of the 'mother' function $\Xi(\kappa)$ and its derivatives. This property might be accidental, but we have reasons to suspect that, at least for QED in the quenched approximation, it will be found to extend to arbitrary loop orders.

Moreover, we have seen that even in QCD the same would hold true at one loop if a gluon mass was introduced (say, as an infrared regulator). It would be very interesting to know the form of the result for the nonabelian case at two loops, i.e. the two loop effective lagrangian for Yang-Mills theory in a constant self-dual quasi-abelian background. The two-loop worldline formulas derived in [29] for the general nonabelian Euler-Heisenberg type lagrangian should provide a good starting point for its computation.

Our results also suggest a new approach to the computation of the lagrangians $\mathcal{L}^{(2)}_{\text{scal,spin}}$ for a general constant background. Rather than performing a double expansion in the variables a, b, as has been done in [12, 17], it might be preferable to use an expansion in the variables $H \equiv a + ib$ and $\bar{H} \equiv a - ib$. The integration techniques which we have used here for the self-dual case, corresponding to $\bar{H} = 0$, would in principle also allow

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one to obtain, at any fixed order of this expansion in $\bar{H}^n$, a closed form expression for the coefficient as a function of H. This might lead to progress even in the computation of the magnetic and electric cases.

We anticipate that the techniques developed here may also help to extend the one-loop instanton background calculations [30, 31, 32] to the two-loop level.

In part 2 of this paper [54] we investigate the structure of the imaginary part of the two-loop self-dual Euler-Heisenberg lagrangians for the 'electric' case. We also study the weak- and strong-field expansions of these two-loop effective lagrangians, and use them to test the techniques of Borel summation, and the associated analytic continuation properties.

Acknowledgments

We thank Zvi Bern and Gregory Mahlon for correspondence concerning helicity structures in massive loop amplitudes. Discussions and correspondence with Vladimir Ritus and Arkady Tseytlin are gratefully acknowledged. Both authors thank Dirk Kreimer and the Center for Mathematical Physics, Mathematics Department, Boston University, for hospitality during the final stage of this work. We gratefully acknowledge the support of NSF and CONACyT through a US-Mexico collaborative research grant, NSF-INT-0122615.

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